Observations on distorted turbulent wakes

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(Received 3 November 1961)

The question considered here is whether turbulent wakes can undergo a simple self-preserving development when convected through a duct of varying cross-section. A flow is described as self-preserving if the distributions of its mean-value properties at successive sections have the same basic shape, differing only in magnitude and lateral extent. In such flows the scales of the mean velocity variation and turbulence have been found to be proportional.

The wakes studied are concentrated near a plane of symmetry through a duct of changing rectangular section but constant area; the portion of the wake near the other plane of symmetry is considered. Measurements in the flows behind three circular cylinders reveal a lack of universality of the scales associated with self-preserving solutions of the momentum equation. The wakes of the smaller cylinders adopt the predicted form, but that of the largest does not. As it moves through the channel, this wake is compressed less rapidly by the imposed lateral strain than is predicted.

Interest is then concentrated on the equation governing the kinetic energy of the turbulence and in particular on the relative size of its two production terms, that representing the action of the mean wake shear and that representing the effect of the distortion applied by the duct. It is found that self-preserving wakes can form only in a few kinds of distorting convecting flow. In the experimental duct, which has exponentially varying walls, one of these is set up. Attention is concentrated on this case. A stability analysis (based on the criterion that a stable wake is one in which the total turbulence production decreases as a result of more rapid wake expansion) suggests that, when production by wake shear is greater than that by distortion, a stable self-preserving development is possible in which the turbulence scale and the velocity defect scale remain proportional. But in the alternative, distortion-dominated, case the velocity defect decreases continually relatively to the turbulence.

The experimental results are in accord with these predictions. The transition from shear-dominated to distortion-dominated wakes appears to take place when the production terms are about equal. The failure of the one experimental wake to adopt a simple self-preserving form can be attributed to the relatively slight organization of its turbulence when distortion is begun.

1. Introduction

Several years ago Townsend (1954) studied the effect of uniform distortion on homogeneous turbulence by allowing the turbulence generated by a grid to pass through a channel whose width and height varied while the cross-sectional area and hence the convective velocity remained essentially constant. In the present studies turbulent wakes were generated upstream of the distorting section and the forms they adopted in it were examined. Such a flow is not solely determined by an initial impulse applied when it is set up, as are many free turbulent flows, but may be influenced also by the subsequent straining of the convecting stream. It is then of particular interest to inquire whether there is possible in this more complicated situation a simple self-preserving development like those of other free turbulent flows. For comparison, the basic results of a search for self-preserving forms will be presented before we turn to the experimental results.

2. Velocity and width scales for self-preserving distorted wakes

We shall consider wakes introduced into a stream which when undisturbed is described by

$$U = U_1, \text{ a constant,}$$
$$V = -a'(x)\,U_1 y,$$
$$W = a'(x)\,U_1 z.$$

The streamlines of this flow are given by

$$y \propto e^{-a(x)}, \quad z \propto e^{a(x)}.$$

A close approximation to the basic pattern should be established in a constant-area duct with walls defined by equations of this form, provided that $a'(x)$ is sufficiently small and slowly varying and that compensation is made for boundary-layer growth.

The wake will be taken to be concentrated near the plane $y = 0$; we shall consider the portion near the other plane of symmetry, $z = 0$. For distorting flows that either convect the wake towards the plane $y = 0$ ($a'(x) > 0$) or, at worst, do not expand it too violently, the 'boundary-layer' approximation will still be valid. Proceeding on this basis we obtain from

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial \overline{u^2}}{\partial x} + \frac{\partial \overline{uv}}{\partial y} = -\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right),$$

(valid near the plane of symmetry $z = 0$) the simplified form

$$U_1\left[\frac{\partial U}{\partial x} - a'(x)\,y\frac{\partial U}{\partial y}\right] + \frac{\partial \overline{uv}}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2}. \tag{1}$$

A momentum integral can be obtained by integrating over two planes perpendicular to the x-axis and over a suitable stream surface joining them:

$$U_1 \int_0^{e^{a(x)}} dz \int_{-Be^{-a(x)}}^{Be^{-a(x)}} (U_1 - U)\,dy = \text{constant},$$

or

$$e^{a(x)} \int_{-Be^{-a(x)}}^{Be^{-a(x)}} (U_1 - U)\,dy = \text{constant}, \tag{2}$$

where the constant B must be chosen large enough to extend the integration beyond the wake but not so large that the integrated effect of the non-uniform pressure is comparable with the momentum flux. These conflicting conditions cannot both be satisfied over a very large range of x. In deriving this relationship we have assumed the mean streamlines to retain their undisturbed shape; this is consistent with the resulting momentum integral which indicates that the mass flux defect varies as $\exp\{-a(x)\}$, just as do the undisturbed streamlines in the (x,y)-plane.

(a) Asymptotic scales for turbulent distorted wakes

Following Townsend (1956) we seek solutions to the momentum equation in the form

$$\left.\begin{aligned} U &= U_1 + u_0 f(\eta) \\ \overline{uv} &= u_0^2 g_{12}(\eta) \end{aligned}\right\} \eta = y/l_0, \tag{3}$$

where $u_0(x)$ and $l_0(x)$ are velocity and width scales whose variation specifies the basic geometry of the motion and $f(\eta)$ and $g_{12}(\eta)$ are the self-preserving forms for the variations of mean velocity and mixing stress across the wake.

From equations (1) and (2) we find

$$U_1\left[\frac{l_0}{u_0^2}\frac{du_0}{dx}f - \frac{\eta}{u_0}\left(\frac{dl_0}{dx}+\frac{da}{dx}l_0\right)f'\right] + g_{12}' = \frac{\nu}{u_0 l_0}f''$$

and

$$e^{a(x)}u_0 l_0 \int_{-B\exp[-a(x)]/l_0}^{B\exp[-a(x)]/l_0} f(\eta)\,d\eta = \text{constant.}$$

The integral is in fact independent of x so long as the range of integration extends beyond the wake at every station. If these relations are to be of self-preserving form we must have

$$\left.\begin{aligned} \frac{l_0}{u_0^2}\frac{du_0}{dx} &= -\frac{c_1}{U_1}, \quad \text{a constant,} \\ \frac{1}{u_0}\left(\frac{dl_0}{dx}+\frac{da}{dx}l_0\right) &= \frac{c_2}{U_1}, \quad \text{a constant,} \\ \text{and} \qquad e^{a(x)}u_0 l_0 &= \text{constant.} \end{aligned}\right\} \tag{4}$$

In general the viscous term cannot be retained in the self-preserving equation. For the contracting case of special interest the importance of the viscous term increases downstream; ultimately wakes of this class may cease to be turbulent. Note also that the second condition for self-preservation admits of a simple interpretation: the net wake spread is, in a self-preserving wake, the sum of the imposed convection at the boundary and an expansion proportional to the turbulence scale.

The scales given by (4) are

$$\left.\begin{aligned} l_0^2 e^{2a(x)} &\propto \int^x e^{a(x')}dx' + \text{constant,} \\ \frac{1}{u_0^2} &\propto \int^x e^{a(x')}dx' + \text{constant.} \end{aligned}\right\} \tag{5}$$

If $a'(x) = a$, a positive constant, the streamlines are given by $y \propto e^{-ax}$. Asymptotically,

$$l_0 \propto e^{-\frac{1}{2}ax}, \quad u_0 \propto e^{-\frac{1}{2}ax}.$$

Again, if $a'(x) \propto 1/x$, the streamlines are given by $y \propto 1/x^m$, and, asymptotically,

for $m > -1$,	for $m < -1$,
$l_0 \propto x^{-\frac{1}{2}(m-1)}$,	$l_0 \propto x^{-m}$,
$u_0 \propto x^{-\frac{1}{2}(m+1)}$;	$u_0 = \text{constant}$.

Very likely the results for $m < -1$ are not physically meaningful: clearly the assumptions about the flow geometry are violated.

Returning now to the self-preserving form of equation (1) we find that it may be written

$$c_1 \eta f = g_{12}$$

since the momentum invariant indicates that $c_1 = c_2$. This form is a convenient starting-point for an analysis based on some theory of the structure of the turbulence in the wake.

The self-preserving forms discovered here for distorted wakes cannot be expected to be as comprehensive as those for the simple wake. The neglect of pressure terms in deriving the momentum integral must inevitably limit its applicability to only a portion of the wake. Moreover, for the contracting case of prime interest, the importance of the viscous term may increase until the turbulence is no longer maintained.

(b) *The asymptotic forms of laminar distorted wakes*

Proceeding as before we find that

$$\frac{l_0^2}{u_0}\frac{du_0}{dx} = -\frac{c_3}{U_1}, \quad \text{a constant,}$$

$$l_0\left(\frac{dl_0}{dx} + \frac{da}{dx} l_0\right) = \frac{c_4}{U_1}, \quad \text{a constant,}$$

$$e^{a(x)} u_0 l_0 = \text{constant,}$$

must be satisfied by self-preserving scales. Again we have

$$\frac{u_0 l_0}{\nu} = e^{-a(x)},$$

indicating that laminar flow is sustained in a compressed wake. We shall consider the same examples as for the turbulent case. If $a(x) = ax$, $a > 0$, then asymptotically $l_0 = \text{constant}$, $u_0 \propto e^{-ax}$. Or if $e^{-a(x)} \propto x^{-m}$, then, asymptotically,

for $m > -\frac{1}{2}$,	for $m < -\frac{1}{2}$,
$l_0 \propto x^{\frac{1}{2}}$,	$l_0 \propto x^{-m}$,
$u_0 \propto x^{-\frac{1}{2}(2m+1)}$;	$u_0 = \text{constant}$.

The behaviour of these wakes is very different from that of the corresponding turbulent wakes. This may be rationalized by comparing the second condition on these scales with the corresponding condition on the scales for turbulent wakes.

The self-preserving form of the momentum equation can now be integrated (here $c_3 = c_4$) to give

$$f = f_0 \exp\{-(c_3/2\nu)\eta^2\}.$$

3. An experimental study of width and velocity scales

Townsend's measurements of simple turbulent wakes behind circular cylinders have shown that the mean velocity profiles can adopt an approximately self-preserving form for $x/d = 80$, long before the various measures of turbulent intensity do so. Thus the determination of velocity and width scales characterizing the velocity defect should be the most convenient means of testing the predictions of the similarity analysis. The peak velocity defect and the width at half the peak value are the quantities that will be taken as velocity and length scales.

In order to make clear the departure of the actual flows from full self-preservation, comparisons will be made with two mathematical models, one representing a self-preserving wake in an undisturbed stream, the other a wake in a distorting flow modelling that of the experimental situation. The second model wake is in the condition of undistorted self-preservation on reaching the distorting duct and retains a self-preserving form while moving through it. The derivation of these models is relegated to an appendix.

(a) *Experimental apparatus*

The distorting duct used in these studies was that used by Townsend (1954) in his investigation of the uniform distortion of homogeneous grid turbulence. Once again this duct formed part of the open-return wind tunnel described by Townsend, only two modifications being made to the original arrangement: a wooden former $1\frac{1}{2}$ in. wide was fitted in the slot for grids just downstream of the contraction, and a more rigid section, 18 in. long, was introduced after the distorting duct. On this added section was mounted a stout traversing gear with which a longitudinal rod carrying sensing instruments could be deflected to traverse the instruments vertically across the duct.

The wakes studied were those behind circular cylinders inserted through the former mentioned above. Thus the distorting section began $20\frac{3}{4}$ in. downstream of the cylinder axis and extended for 40 in. to the parallel-sided recovery section. The distorting duct had exponentially shaped sides, its section changing from height 24 in., width 6 in. to height $6\frac{1}{4}$ in., width $24\frac{1}{4}$ in. over the 40 in. length. The slight increase in area compensated for boundary-layer growth. The cylinders were introduced horizontally; the convection induced by the duct walls was inwards towards the plane of the wake. In the derivation of the self-preserving models the convection has been specified by

$$\left.\begin{aligned} a(x) &= \frac{a}{2(x_2-x_1)}(x-x_1)^2 \quad \text{for} \quad x_2 > x > x_1, \\ &= ax - \frac{a}{2}(x_2+x_1) \qquad \text{for} \quad x > x_2, \end{aligned}\right\} \tag{6}$$

with $a = 0{\cdot}0346\,\text{in.}^{-1}$, $x_2 - x_1 = 20\,\text{in.}$ This corresponds to a graded imposition over a 20 in. section of the duct, as suggested by the results of Townsend (1954). (Equation (6) has been given in the appendix as equation (A 2).)

(b) *Velocity defect measurements*

A small pitot-tube was used to determine the velocity distribution at seven stations of the distorted wake behind a cylinder of diameter $\frac{1}{2}$ in. Two difficulties were encountered: the same free-stream pressure was not found on both sides of the wake, indicating a lack of uniformity in the stream approaching the cylinder, and considerable drift was encountered in the sensitive manometer, presumably due to temperature variation. The second problem was overcome by taking several sets of readings at each station, the first by subtracting from the averaged readings a linear function of lateral position.

$s = x - x_1$ (in.)	$\frac{b}{d}$	$\frac{u_m}{U_1}$	$e^{a(s)}$	$e^{a(s)}\frac{bu_m}{dU_1}$
5	2·69	0·1472	1·022	0·405
10	3·26	0·1228	1·090	0·437
15	3·14	0·1143	1·215	0·436
25	3·04	0·0880	1·680	0·455
35	2·77	0·0736	2·377	0·485
45	2·43	0·0590	3·36	0·482
50	2·56	0·0515	—	—

TABLE 1. Velocity defect data.

From the plotted results were obtained u_m, the peak velocity defect, and b, the width of the distribution for which the defect had fallen to half its peak value. These quantities are given in table 1. From these data has been computed the momentum invariant for a self-preserving wake under the idealized distortion (6). The values in the last column of table 1 may be compared with those from Townsend's measurements of an undisturbed wake:

$$\frac{bu_m}{dU_1} = 0{\cdot}512 \quad \text{for} \quad x/d = 80 \text{ to } 160,$$
$$= 0{\cdot}39 \quad \text{for} \quad x/d = 500 \text{ to } 950.$$

In the present tests, $x/d = 30$ (for $s = 5$ in.) to 110 (for $s = 45$ in.).

The peak velocity is plotted in figure 1 together with curves representing self-preserving models (from equations (A 3), (A 8) of the appendix, and results similar to equations (A 6), (A 7)). Since the distortion is initiated at $x/d \simeq 25$ for the wake behind the $\frac{1}{2}$ in. cylinder, this wake cannot be expected to be of self-preserving form at the beginning of the distortion; it will still be expanding more rapidly than is indicated by the self-preserving scale. It is not surprising then to find that the peak velocity defect is initially greater than that predicted by the similarity law, while subsequent measurements fall below the model values.

In figure 2 the velocity defect measured at one section of the distorted wake is plotted with a normal curve enclosing the same area. The normal distribution gives a very good representation of the mean velocity variation, as is the case in an undistorted wake, but as there the measured values fall more rapidly to zero at the edge of the wake.

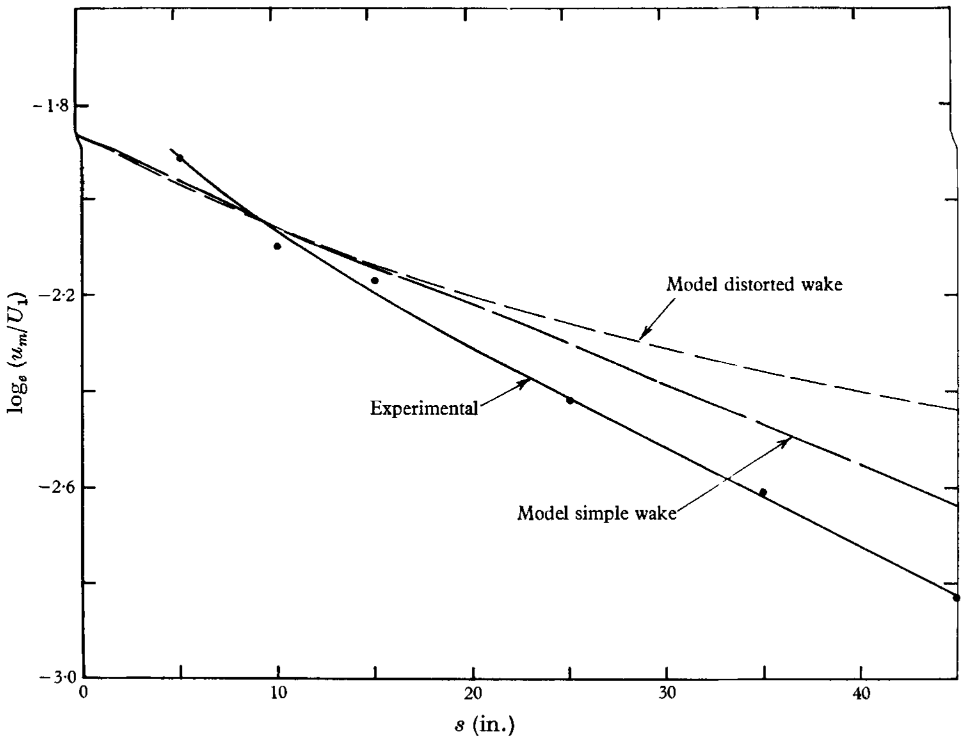

FIGURE 1. Variation in peak velocity defect in wake behind ½ in. cylinder.

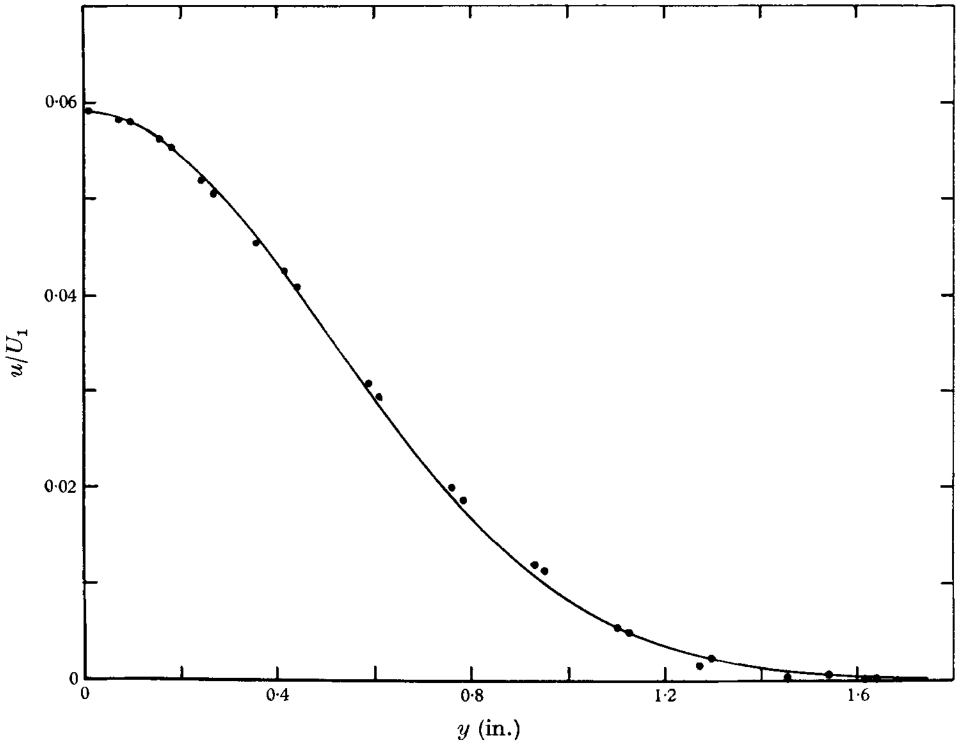

FIGURE 2. Comparison of measured velocity defect to normal distribution (s = 35 in.).

model suggests. Also, the wake of the largest cylinder does not conform to the self-preserving pattern anywhere in the distorting duct. In this wake neither the variation of width nor the structure of the turbulence is that associated with self-preservation. These deviations suggest immediately that wakes that have adopted

d (in.)	Reynolds number	x/d for imposition of distortion
$\frac{3}{16}$	2000	57 to 164
$\frac{5}{16}$	3400	37 to 92
$\frac{1}{2}$	5300	21 to 62
	4900 (for velocity defect measurements)	

TABLE 2. Test data.

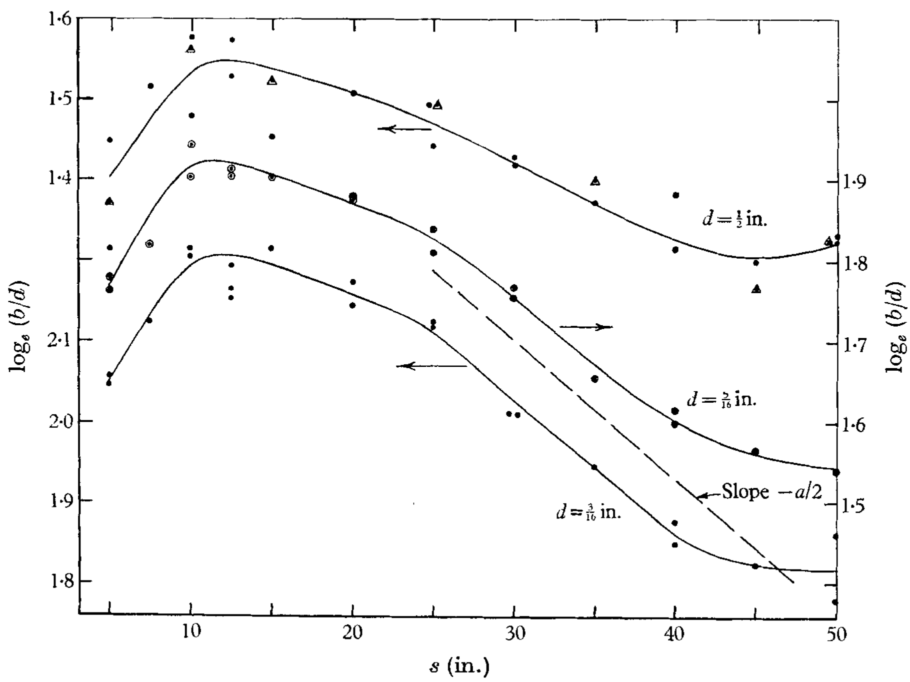

FIGURE 3. Variation of width of wakes behind circular cylinders of diameters $\frac{3}{16}$, $\frac{5}{16}$, and $\frac{1}{2}$ in. The points marked △ represent velocity defect measurements with a translated vertical scale.

one self-preserving form (or have gone some way towards doing so) are able to move much more rapidly into another self-preserving mode than are those in which large departures from self-preservation exist initially. In the following discussion it will be seen to what extent this premise is justified. But before examining the factors that decide whether self-preservation will be achieved, we shall consider the reaction of a wake during the early stages of straining. The model suggests. Also, the wake of the largest cylinder does not conform to the self-preserving pattern anywhere in the distorting duct. In this wake neither the variation of width nor the structure of the turbulence is that associated with self-preservation. These deviations suggest immediately that wakes that have adopted

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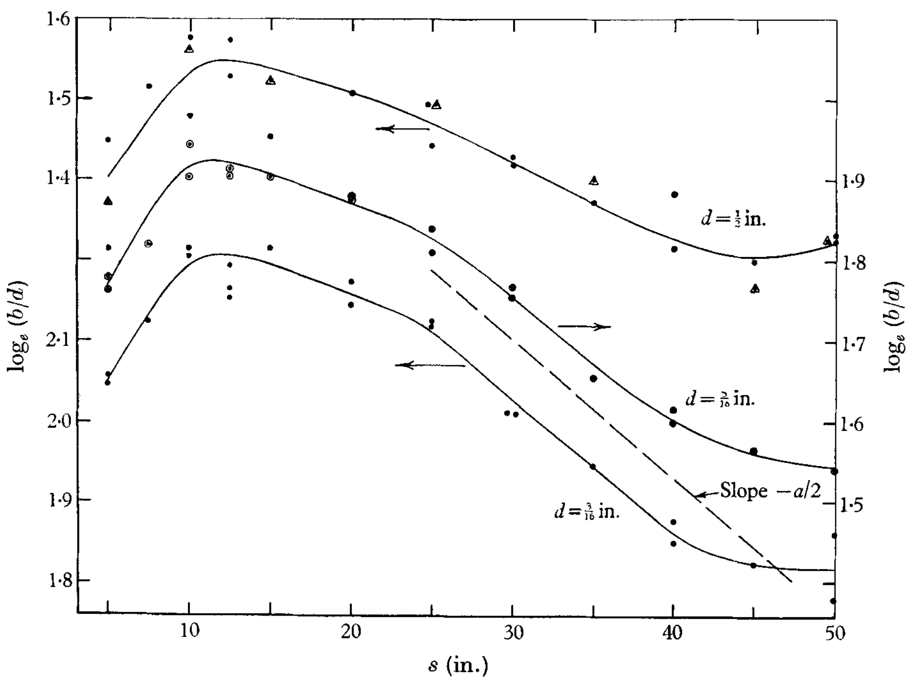

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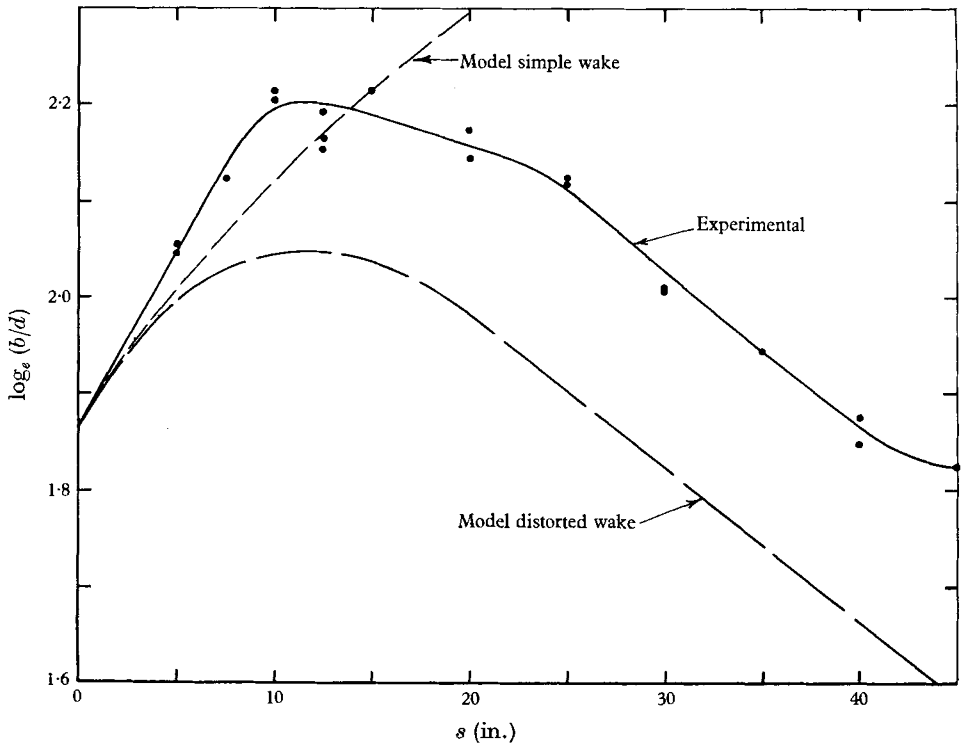

FIGURE 4. Comparison of model wakes with measurements for $\frac{3}{16}$ in. cylinder.

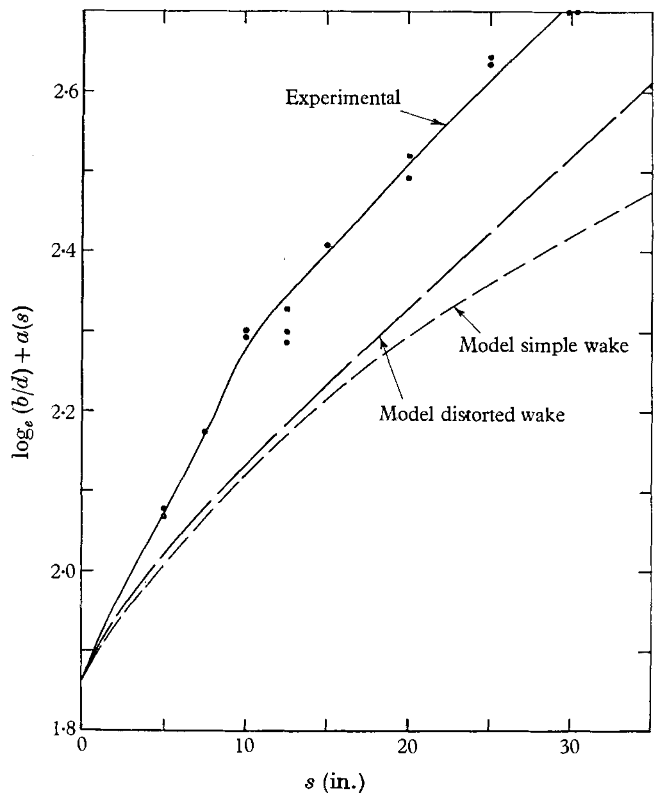

FIGURE 5. Comparison of wake spreading relative to mean streamlines for $\frac{3}{16}$ in. cylinder.

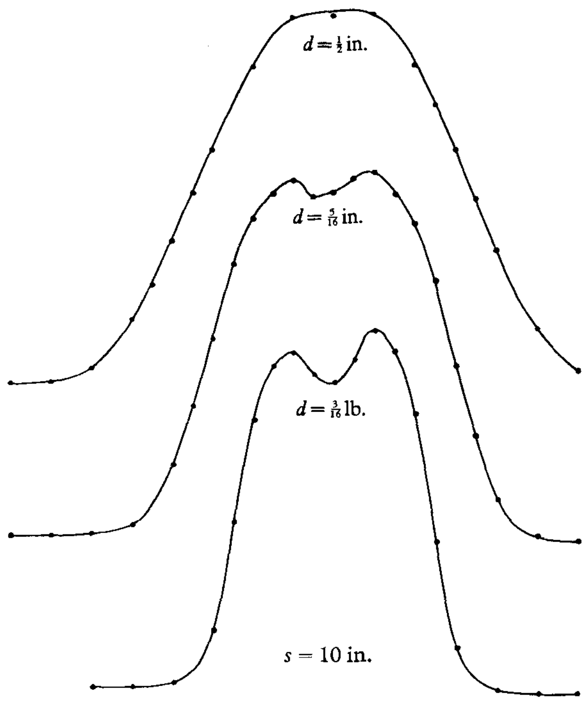

FIGURE 6

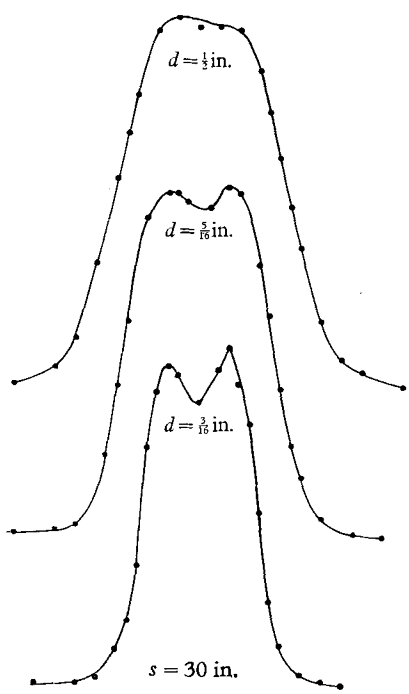

FIGURE 7

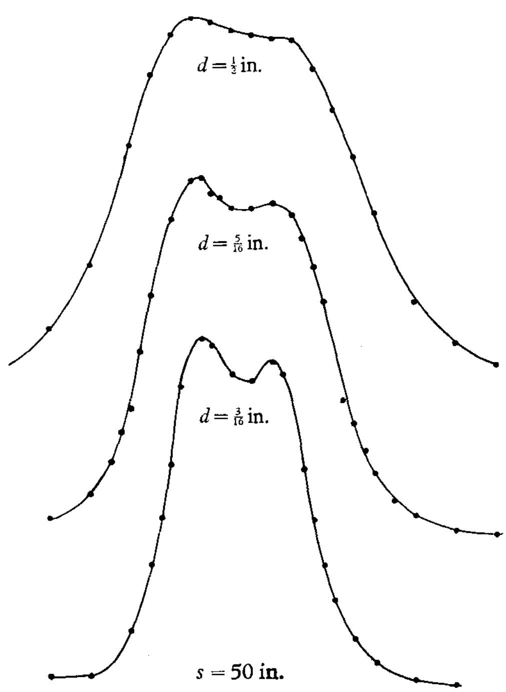

FIGURE 8

FIGURES 6–8. Profiles of intensity of streamwise component of turbulence ($\overline{u^2}$ *vs* y) at entrance to distorting duct (s = 10 in.), half way through it (s = 30 in.), and at its end (s = 50 in.) for $\frac{3}{16}$, $\frac{5}{16}$ and $\frac{1}{2}$ in. cylinders. Scales arbitrary.

nature of the departures from self-preservation will be explained and the possibility of a rapid approach by the turbulence to an equilibrium structure will be discussed.

(*a*) *Initial rapid expansion*

The basis of the similarity postulates of § 2 (equations (3)) was the assumption that the scale of the velocity defect was proportional to the scale of the turbulence so that the flow could be characterized by a single velocity scale. The justification for the linking of the scales of turbulence and mean velocity variation is that the shear associated with the latter is the sole means of maintaining the turbulence of a wake once the budget of turbulence energy introduced near the generating body has been dissipated. But the interdependence of mean shear and turbulence can be expected to break down if a new mechanism for generating turbulence is suddenly introduced, as is the case in a distorting mean flow.

Townsend's study of the distortion of homogeneous turbulence shows (Townsend 1954, figures 3 and 4) that the component of turbulence in the direction of compression is greatly augmented relative to the unstrained state and to the other components. And it is this component that may be expected to be chiefly responsible for the spread of the turbulent region. Grant's (1958) studies of wakes revealed large turbulent motions in the form of laterally directed jets which had an important part in wake expansion.

We can now see how the assumed proportionality of velocity scales breaks down. The amplification of the wake turbulence (and especially of its lateral component) by the added strain gives rise to a rapid lateral expansion of the turbulence wake. The velocity defect wake cannot lag far behind in this expansion; the comparison of wake widths based on defect and turbulence (figure 3) shows that it does not. Thus the increase in the turbulence scale (relative to that in an undisturbed wake) must be coupled with a relative reduction in the scale of the velocity defect. The variation of velocity defect shown in figure 1 is consistent with this prediction.

(*b*) *Abrupt end to expansion*

The initial period of rapid expansion appears (figure 4) to end quite abruptly when the rate of strain has (presumably) achieved about one-half its ultimate value; the measured wake width is thereafter nearly proportional to the self-preserving model. Before considering possible causes for the termination of this rapid growth, we should note that the break in the expansion does not appear so extreme in figure 5 where account is taken of inwards convection.

A possible explanation for the sharp break in wake growth is that it marks an approach to an equilibrium structure in the strained turbulence. In such a structure the lateral component, that chiefly responsible for wake spreading, could no longer grow independently of the other components; some of the energy added to it would be distributed between the others. The idea that the turbulence structure may be near equilibrium when the rate of strain has not attained its maximum value and when the total strain is still small seems to be contradicted by the behaviour of the distorted homogeneous turbulence studied by Townsend. There an equilibrium structure was set up only after the strain had been applied for nearly the whole length of the duct. This objection can be countered by observing that the wake turbulence has been given a definite structure by the mean wake strain while the interaction of many wakes had produced near-isotropy in the grid turbulence entering the distorting duct in Townsend's studies. It is consistent with Townsend's postulate of similar equilibrium structures of highly strained turbulence in various flows that the wake structure need change only slightly when a new strain is applied to it. The more general argument that the possible degree of anisotropy must be limited may also be invoked.

Nevertheless, it seems doubtful that an approach to an equilibrium structure is the decisive factor in the reduction of the initial rate of spread. The wake of the $\frac{1}{2}$ in. cylinder appears (figure 3) to react very much as do the other wakes during the initial stages of distortion, while its structure is quite different (figure 6). In the absence of an initially highly structured turbulence we cannot justify such a rapid approach to equilibrium. In this connexion it is not helpful to note that the wakes behind the smaller cylinders quickly assume self-preserving width scales once the rate of distortion has become constant. Wake spread is not an accurate index of the structure of turbulence. This can be seen by considering the wake in simple rectilinear convection. As has been noted earlier, the scales of velocity defect and wake width take up self-preserving forms long before the turbulence does so.

This line of thought suggests another possible explanation of the abrupt end to the initial expansion. Rapid lateral and transverse spreading must reduce the shearing intrinsic to a wake, and thus check turbulence production through this mechanism. The large 'eddies', those chiefly responsible for wake growth, are directly dependent on the mean shear. It is, of course, the largest, most slowly changing eddy forms that are most influenced by a steady distortion. Thus they are amplified also when the external strain is first applied. But they soon break up and their successors, set up by the weakened shear, are somewhat less intense.

We may note too that the first effect of distortion may be to release quickly many of the laterally directed jets found by Grant (1958), thus depleting the number of the special structures from which they might later have been formed. Further, it could be this mechanism, rather than amplification, that is chiefly responsible for the early rapid expansion of a strained wake.

(c) *The competition of strains*

Following the wake a little further into the duct, we next consider the factors that decide whether or not it finally adopts the self-preserving forms of §2. Our considerations will be based on the equation governing the kinetic energy of the turbulence. The approximate form of this equation applicable to simple shear flow in a distorting, constant-area convection is

$$\frac{D}{Dt}(\overline{q^2}/2) = \underset{(a)}{-\overline{uv}\frac{\partial U}{\partial y}} + \underset{(b)}{(\overline{v^2}-\overline{w^2})\,a'(x)\,U_1} - \underset{(c)}{\frac{\partial}{\partial y}[\overline{v(p+q^2/2)}]} - \underset{(d)}{\frac{\nu}{2}\overline{\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]^2}}, \quad (7)$$

with
$$\frac{D}{Dt} = U_1 \left[\frac{\partial}{\partial x} - a'(x)\, y \frac{\partial}{\partial y} \right].$$

As before, we have restricted ourselves to the region near the plane of symmetry $z = 0$.

The terms (a) and (b) represent production of turbulence energy by the two strains applied to the turbulence in this flow; (a) is absent from the equation governing homogeneous distorted turbulence, (b) from that governing an undistorted wake. The term (c) represents the most important processes by which turbulence energy is transmitted laterally through the flow; it too is absent from the equation for homogeneous distorted turbulence. The term (d) represents dissipative processes. It is the relative size of the production terms (a) and (b) that determines the response of a wake to distortion by the basic flow; these are the driving terms of the equation, the others must adopt values consistent with them.

The production term associated with the straining of the mean velocity distribution (a) is independent of the turbulent intensity and of the distortion, except in so far as distortion alters the velocity defect scales. This may be seen as follows. If we assume a scale u_0 for the velocity defect (but not for the turbulence) we obtain
$$\frac{\partial \overline{uv}}{\partial \eta} = -U_1 \left[l_0 \frac{du_0}{dx} f - \eta u_0 \left(\frac{dl_0}{dx} + \frac{da}{dx} l_0 \right) f' \right]$$
$$= -U_1 l_0 \frac{du_0}{\partial x} (\eta f)',$$
on using the momentum equation (1) and integral (2). Then
$$-\overline{uv} \frac{\partial U}{\partial y} = U_1 u_0 \frac{du_0}{dx} (\eta f f'). \tag{8}$$

We have seen (figure 2) that the self-preserving form $f(\eta)$ is not much affected by distortion.

The production term (b) is, on the other hand, dependent ultimately on the intensity of the distorted turbulence. The distortion of originally isotropic turbulence (Townsend 1954) gives rise to a structure such that turbulence production is achieved ($\overline{v^2} > \overline{w^2}$). The turbulence in a moderately well-developed wake is, prior to distortion, of slightly adverse form; that is, $\overline{w^2} > \overline{v^2}$. But the applied strain will quickly modify the structure to achieve a net production of turbulence energy through the processes represented in term (b).

If distortion is applied to a wake whose turbulence has not been given form by the mean shear (the process of term (a)), there is a possibility that the production by distortion (term (b)) will achieve dominance; that the wake will spread so rapidly as to reduce further the importance of shear production; and finally that the turbulence will develop more like strained homogeneous turbulence than wake turbulence. But if the turbulence of a wake is sufficiently well organized, that is, if the decay and development have progressed far enough to make $\overline{uv}$ comparable in magnitude with $\overline{v^2}$, the intrinsic wake shear may maintain control so that the decaying turbulence follows closely the development of the velocity defect. One possible form of interaction between the two kinds of production has already been mentioned, a drop in production by shear (*a*) as a result of augmented wake expansion. There is also the possibility that initially highly structured turbulence is a factor in limiting the anisotropy ($\overline{v^2} > \overline{w^2}$) preferential to production by distortion (*b*).

These ideas are consistent with the evidence of figures 6–8. There we see that the turbulence behind the smaller cylinders has at the entrance to the distorting duct already adopted the structure characteristic of interaction with a mean velocity defect flow, the production concentrated in regions of maximum shear. This structure is maintained in these wakes throughout the distorting duct. On the other hand, the turbulence behind the $\frac{1}{2}$ in. cylinder is still unformed by the mean shear on entering the distorting duct. This intensity profile retains its simple shape right through the distorting section. At no point does production by wake shear have a dominant role in forming the turbulence of this wake.

It may seem puzzling that so marked a change in response can take place over such a small range of cylinder diameter. But the fall in turbulent intensity in the first part of a wake ($x/d < 100$, say) is far more rapid than is suggested by the law

$$\frac{\overline{u^2}}{U_1^2} \propto \left(\frac{x-x_0}{d}\right)^{-1}.$$

(See Townsend 1956, figure 7.5, etc.) Then the intensity of the wake turbulence at $x/d \simeq 30$ where the wake behind the $\frac{1}{2}$ in. cylinder is strained may be greater by a factor of 3 or 4 than that at $x/d \simeq 50$ where the wake of the $\frac{5}{16}$ in. cylinder is distorted.

(*d*) *The criterion for self-preservation*

Finally, an attempt will be made to give a more precise form to the ideas developed above. We take the ratio of the two production terms of equation (7) as a measure of their relative importance, tentatively defining a wake dominated by shear to be one for which

$$R = \frac{-\overline{uv}(\partial U/\partial y)}{(\overline{v^2}-\overline{w^2})\, a'(x)\, U_1} > 1.$$

This criterion can be rewritten

$$A = \frac{-\overline{uv}}{\overline{v^2}-\overline{w^2}} > \frac{a'(x)}{U_1^{-1}(\partial U/\partial y)},$$

in which form it prescribes a degree and kind of organization which must be attained if wake shearing is to remain an important influence in turbulence production. In order to see what level of organization is required, we shall estimate the strain ratio on the right-hand side of this inequality for the wakes studied experimentally, considering their form at the nominal entrance to the distorting duct.

Townsend's measurements of the velocity defects of simple wakes in the range $x/d = 80$ to 160 can be described with good accuracy by

$$\frac{u}{U_1}\left(\frac{x-x_0}{d}\right)^{\frac{1}{2}} = 1{\cdot}00 \exp(-10{\cdot}6\eta^2); \quad x_0 = -20d;$$

with $U = U_1 - u$. Hence we find the maximum velocity gradient in the wake:

$$\frac{1}{U_1}\left(\frac{\partial U}{\partial y}\right)_m = \frac{2\cdot 80}{x - x_0}.$$

To represent a typical value of wake shear and to account for the slight broadening at the entry to the distorting section (figure 4) we introduce a factor $\frac{1}{2}$ in estimating the shear. As the criterion for a shear-dominated wake we have now

$$A > \frac{aU_1}{(\partial U/\partial y)_m},$$

with $(\partial U/\partial y)_m$ given above. In table 3 are given the values of the strain ratio in the wakes behind the three cylinders.

The criteria for the three wakes do not vary greatly. This is due to the weak dependence (through x_0) of the wake shear on the diameter of the generating cylinder. One fact emerges clearly and is not dependent on the particular criterion chosen here: the strains applied in the early stages of distortion are nearly the same for all three wakes. Then it is to the structure of the turbulence that we must ascribe the differences in their reactions to the combined strain.

d (in.)	$\frac{1}{U_1}\left(\frac{\partial U}{\partial y}\right)_m$	$aU_1\Big/\left(\frac{\partial U}{\partial y}\right)_m$
$\frac{3}{16}$	0·115	0·30
$\frac{5}{16}$	0·103	0·34
$\frac{1}{2}$	0·091	0·38

TABLE 3. Strains at entrance to distorting duct.

For the wake of the $\frac{3}{16}$ in. cylinder the criterion suggested for dominance of production by wake shear is $A > 0\cdot 30$. In view of the well-developed structure in this wake, the criterion may well be satisfied even when the added distortion has made considerable progress in changing the structure. But for the $\frac{1}{2}$ in. cylinder wake we must have $A > 0\cdot 38$. Here it seems very doubtful that the correlation $\overline{uv}/[\overline{u^2}\overline{v^2}]^{\frac{1}{2}}$ will be large enough initially to allow the criterion to be satisfied for long as the added distortion proceeds to modify the turbulence.

It appears then that the critical value of the parameter R cannot be very different from the assumed value of unity.

5. The final forms of wakes in distorting flows

In the experimental results and discussion of the preceding sections we have distinguished two kinds of wake development during the period in which the distortion is established. We now consider the forms ultimately adopted by wakes following these two courses.

One development, that revealed by a conventional search for self-preserving solutions, is governed by the production of turbulence by the shear intrinsic to the wake. In its final form (which is probably not attained in the short duct used in the experiments) the scales of velocity defect and turbulence will once again be proportional. The nature of the transition suggests that the ratio of turbulence scale to mean velocity scale will be higher than in an undistorted wake. Certainly the expansion relative to the mean streamlines is more rapid than in that flow (figure 5).

In the alternative wake development the turbulence becomes independent of the mean shear of the velocity defect, being maintained by the strain applied by the distorting duct. This form is not prescribed by a search for self-preserving solutions of the momentum equation (1). Since the Reynolds stress contributing to the momentum balance loses its control over the overall development, there is no longer a link between the velocity defect and the turbulent intensity and the associated rate of wake spread. Then the situation which arose in the initial stages of distortion—an increase of the turbulence scale relative to the velocity defect scale—can be perpetuated.

The consideration of the momentum equation in § 2 (*a*) gave no indication of this second mode of wake development. But the subsequent discussion has hinged on the production of turbulence energy. This suggests that the equation (7) governing the turbulence energy may provide a more satisfactory basis for a study of self-preservation. In contrast, no such 'higher appeal' is possible for the distorted laminar wakes whose self-preservation was considered in § 2 (*b*).

(*a*) *Reconsideration of self-preservation*

As before we assume a self-preserving mean velocity variation

$$U = U_1 + u_0 f(\eta), \quad \eta = y/l_0,$$

with u_0, l_0 the scales of the velocity defect. The term of equation (7) which represents production by mean shear is again

$$-\overline{uv}\frac{\partial U}{\partial y} = U_1 u_0 \frac{du_0}{dx}(\eta f f'). \tag{8}$$

We assume also that the turbulence is self-preserving:

$$\tfrac{1}{2}\overline{q^2} = v_0^2 g_{ii}(\eta), \quad \overline{pv} + \tfrac{1}{2}\overline{q^2 v} = v_0^3 h(\eta),$$

$$\overline{v^2} = 2v_0^2 g_{22}(\eta), \quad \overline{w^2} = 2v_0^2 g_{33}(\eta),$$

$$\frac{1}{2}\overline{\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]^2} = \frac{v_0^2}{l_1^2} n(\eta),$$

where v_0 is the turbulence velocity scale and l_1 is a length characterizing the decay of the turbulence.

Equation (7) can now be written

$$U_1\left[2v_0\frac{dv_0}{dx}g_{ii} - \eta\frac{v_0^2}{l_0}\left(\frac{dl_0}{dx} + \frac{da}{dx}l_0\right)g'_{ii}\right]$$
$$= U_1 u_0 \frac{du_0}{dx}\eta f f' + 2\frac{da}{dx}v_0^2 U_1(g_{22} - g_{33}) - \frac{v_0^3}{l_0}h' - \nu\frac{v_0^2}{l_1^2}n(\eta). \tag{9}$$

Case I. Production by wake shear remains important

In this case $v_0 \propto u_0$ and the conditions (4) derived from the momentum equation are still relevant. The energy equation becomes

$$c_1[\eta g'_{ii} + 2g_{ii}] = \underset{(a)}{g_{12}f'} - \underset{(b)}{2\frac{da}{dx}\frac{l_0}{u_0}U_1(g_{22}-g_{33})} + \underset{(c)}{h'} + \underset{(d)}{\frac{\nu l_0}{v_0 l_1^2}n(\eta)}.$$

The convection terms on the left-hand side and the terms (a) and (c) are now rendered independent of x. But the term (b) representing the activity of the imposed distortion is not of the required form for an arbitrary choice of $a'(x)$. The general form† of the distortion for which this energy equation is rendered self-preserving is

$$a'(x) = \left(\frac{x}{m}+\frac{1}{a}\right)^{-1}.$$

Thus only the distortions specifically treated in §2 are compatible with self-preservation of a turbulent wake. The dissipation term (d) represents passive processes and must model itself on the others with

$$l_1 \propto l_0\left(\frac{u_0 l_0}{\nu}\right)^{-\frac{1}{2}}.$$

Case II. Production by distortion becomes dominant

Here the term (a) ultimately becomes negligible as $u_0/v_0 \to 0$. Equation (9) can be written

$$U_1\left[\frac{2}{(da/dx)\,v_0}\frac{dv_0}{dx}g_{ii} - \frac{\eta}{(da/dx)\,l_0}\left(\frac{dl_0}{dx}+\frac{da}{dx}l_0\right)g'_{ii}\right]$$
$$= 2U_1(g_{22}-g_{33}) - \frac{v_0}{(da/dx)\,l_0}h' - \frac{\nu}{(da/dx)\,l_1^2}n(\eta).$$

This equation is of self-preserving form if

$$v_0^p \propto e^{-a(x)}, \quad l_0^q \propto e^{-a(x)}, \quad \exp\left\{-\left(\frac{1}{p}-\frac{1}{q}\right)a(x)\right\} \propto a'(x), \tag{10}$$

with p and q constants. The last condition shows that self-preservation is possible again only for

$$a'(x) = \left(\frac{x}{m}+\frac{1}{a}\right)^{-1}.$$

We have not yet made use of the momentum integral relation (2). It must be used with caution here, for in its derivation a term of form $e^{a(x)}l_0 v_0^2$ (representing the normal Reynolds stress) was neglected, while a term of form $e^{a(x)}l_0 u_0$ was retained. Thus several subcases emerge. In what follows we take $u_0^r \propto e^{-a(x)}$.

† This form is found by combining the conditions (4) with the condition that the coefficient of term (b) be constant.

Case IIa. $v_0^2/u_0 \to 0$

Ultimately, $e^{a(x)}l_0 u_0 =$ constant, as before. Then we have the condition

$$\frac{1}{q}+\frac{1}{r} = 1.$$

But

$$v_0^2/u_0 \propto \exp\left\{\left(-\frac{2}{p}+\frac{1}{r}\right)a(x)\right\},$$

so that $2r > p$.

Case IIb. $v_0^2/u_0 \to$ *constant*

Here $2r = p$, so that $r = \frac{3}{2}$, $p = 3$.

Case IIc. $v_0^2/u_0 \to \infty$

Here ultimately $e^{a(x)}l_0 v_0^2 =$ constant, giving the condition $1/q+2/p = 1$. Also $2r < p$.

Let us next apply these results to the first example of § 2 (*a*) which provides a model of the experimental situation. From the last of conditions (10) we see that $p = q$. Further, in all the cases enumerated above the mean velocity scale drops at least as rapidly as the turbulence scale. Then $p = q > 2 > r$. Hence:

Case	$p = q$	r
I	2	2
II*a*	$< 3, > 2$	$< 2, > \frac{3}{2}$
II*b*	3	$\frac{3}{2}$
II*c*	3	$< \frac{3}{2}$

We may compare these results with experimental values obtained from figures 1 and 3, namely $r = 1{\cdot}69$, $q = 3{\cdot}36$. Note that $q^{-1}+r^{-1} = 0{\cdot}89$ and $q/r = 1{\cdot}99$ for these values.

Case III. Production by shear dominant

Here the energy equation is as for Case I, but the term (*b*) is negligible. The absence of this term does not alter the predicted pattern for self-preservation from that in which the two kinds of production are both active. There are no self-preserving wakes corresponding to the condition $r > 2 > q$.

This is the case governed by the treatment of § 2 which was based entirely on the momentum equation, without consideration of the turbulent energy balance. The remarkable aspect of the experimental results is that the scales characteristic of negligible distortion persist so far into the range where distortion is important. A possible explanation is that a more rapid wake spread would lead to a reduction in the total turbulent energy production, an effect almost certainly inconsistent with more rapid spreading of the turbulent region. We investigate this matter in the following stability analysis.

(*b*) *Stability of self-preserving wakes*

We confine ourselves to the case $a'(x) = a$, a constant, and consider the stability in the régime of Case II*a* of a wake whose development is specified by the scales

$$u_0^r \propto e^{-ax}, \quad l_0^q \propto e^{-ax}, \quad v_0^p \propto e^{-ax}, \quad p = q, \quad r^{-1}+q^{-1} = 1,$$

with respect to other such forms. We define

$$E = E_a + E_b = U_1[u_0(du_0/dx)\,\eta ff' + 2av_0^2(g_{22}-g_{33})]$$
$$= aU_1[-r^{-1}K_a\, e^{-2ax/r}\eta ff' + K_b\, e^{-2ax/q}(g_{22}-g_{33})]$$

and $R = E_a/E_b$ as before. K_a and K_b are constants of proportionality. We find that

$$\frac{dE}{dr} = \frac{2ax}{r}E_b\left[\left(1-\frac{r}{2ax}\right)R-1\right].$$

For $dE/dr > 0$, we may expect the flow to be stable with respect to other flows of the prescribed form. An increase in r gives an increase in E and thus an augmented wake spread. This increase implies a higher value of q and a corresponding, stabilizing decrease in r. We have stability here for

$$R > \left(1-\frac{r}{2ax}\right)^{-1},$$

that is, for $R > 1$ with x large. Also, $dR/dx \propto -(r^{-1}-q^{-1})$, so that $dR/dx \leqslant 0$ for $r \leqslant 2$.

In the régime of important production by wake shear ($R > 1$) we have ultimately a stable self-preserving solution for $r = 2$, unstable solutions for $r < 2$, and of course no solutions for $r > 2$. On the other hand, for the case of important production by distortion ($R < 1$), the flows considered are ultimately unstable for all r. However, solutions with $r < 2$ are stable further downstream than those with $r = 2$.

But the criterion of instability $dE/dr < 0$ ceases to be useful as $r \to \frac{3}{2}$ to give Case IIb. We have seen that $q = 3$ for all $r \leqslant \frac{3}{2}$. This suggests that the development of Case II (with $a'(x) = \text{constant}$) will ultimately proceed with the scales given under Case IIb. The experimental values $r = 1{\cdot}69$, $q = 3{\cdot}36$, $q/r = 1{\cdot}99$ do not provide a decisive verdict.

Note that $E_a, E_b = E_a(\eta), E_b(\eta)$. The stability of the entire wake must depend upon some mean value of R across the wake. In choosing the criterion for table 3 an attempt was made to select such a typical value. It will be remembered that on this basis the critical value of R was estimated to be near unity.

A similar stability analysis may be applied to a wake distorted as in the second example of § 2 (a). From the conditions (10) we find

$$v_0^p \propto x^{-m}, \quad l_0^q \propto x^{-m}, \quad p^{-1}-q^{-1} = m^{-1}.$$

Take $u_0^r \propto x^{-m}$. For stability in the régime of Case IIa it is required that

$$R > \left(1+\frac{xr}{2m}\right)^{-1}.$$

For $r/m > 0$, stability is assured ultimately (for large x), as might be expected. For the case $m \to 0$ (simple rectilinear convection) the self-preserving wake is stable for all x.

(*c*) *The ultimate forms of distorted wakes*

We consider now the strains in the wakes studied experimentally in what appear to be their ultimate forms of development. For the central portion of the distorting duct the data of figure 3 can be described by the relationships

$$b_T/d = A\,e^{-as/q},$$

where the constants A are characteristic of the width of the turbulence wakes. Relationships of similar form will describe the velocity defect width (b_0), but the constants A must be replaced by other constants A', which may be estimated as

$$A' = \frac{0{\cdot}512}{0{\cdot}732}A.$$

(See equation (A 3).) We take $2u_m/b_0$ to represent the wake shear, and using the relation

$$e^{a(s)}\frac{b_0}{d}\frac{u_m}{U_1} = 0{\cdot}48$$

(the constant derived from table 1), we find

$$\frac{2u_m}{b_0 U_1} = \frac{0{\cdot}96}{dA'^2}\exp\left\{-a\left(\frac{1}{r}-\frac{1}{q}\right)s\right\}.$$

It is now possible to determine $2u_m/ab_0U_1$, the ratio of the representative strains. The results are given in table 4.

d (in.)	A'	$\frac{1}{r}-\frac{1}{q}$	$\frac{2u_m}{aU_1b_0}$
$\frac{3}{16}$	8·95	0	1·844
$\frac{5}{16}$	6·84	0	1·894
$\frac{1}{2}$	3·95	1/3·40	$3{\cdot}56\,e^{-as/3{\cdot}40}$ $= 2{\cdot}48$ for $s = 35$ in.

TABLE 4. Strain ratio during distortion.

Throughout the duct the wake shear is largest in the wake of the $\frac{1}{2}$ in. cylinder, even though this shear no longer controls the development and must eventually diminish in importance. This indicates a smaller lateral spread by this wake in the early stages of distortion and thus gives support to the contention (of § 4 (*b*)) that the establishing of an equilibrium structure of turbulence is not an important factor in checking the wake growth soon after distortion is first applied. The evidence of table 4 reinforces also the view that it is the organization of the turbulence that is of prime importance in deciding the course of development rather than the relative magnitude of the two strains.

The near equality of the strain ratios in the wakes of the smaller cylinders gives rise to some interesting speculations. A unique ultimate form for the strain would be quite consistent with the single course of development ($r = 2$) encountered in wakes that do not become distortion-dominated. A unique structure might also be expected in the turbulence.

Whether or not these suppositions are correct, it seems doubtful that there exist ultimate forms of wake development (for $a'(x) =$ constant) in which the turbulence production by distortion is negligible (Case III). A vigorous wake with initially large shear will expand so much before adopting the self-preserving pattern that the shear will be considerably reduced in importance. On the other hand, a weaker wake will be rapidly compressed so that its shear strain becomes more important.

Unfortunately, it is not possible now to do more than speculate about the reaction of a wake in the interesting limiting cases. In the experiments discussed here the two strains were (as is shown in tables 3 and 4) of comparable size and the initial structure of the turbulence was of crucial importance in deciding the mode of development of a distorted wake.

Thanks are due to Dr A. A. Townsend with whom the experiments were planned. The pervasive influence of his earlier work must be so obvious that no further acknowledgement is necessary.

Appendix. Model self-preserving wakes

We consider wakes that are in the undistorted self-preserving condition on reaching the distorting section, requiring that they remain self-preserving throughout and that their width and velocity scales be continuous at the section (x_1) where the distortion is begun. Then if

$$b = b'(x-x_0)^{\frac{1}{2}}$$

specifies the wake width variation prior to distortion, its variation under distortion is found from equation (5) to be

$$\left(\frac{b}{b'}\right)^2 = \exp\{-2[a(x)-a(x_1)]\}\left[\int_{x_1}^{x} \exp[a(x')-a(x_1)]\,dx' + (x_1-x_0)\right]. \quad \text{(A 1)}$$

The influence of the distorting section cannot be imposed instantaneously; an extended transition must be expected instead. Townsend's measurements of mean velocity along the duct centre-line (Townsend 1954, figure 2) suggest that this transition extends fully 10 in. upstream and downstream of the beginning of the distorting section. His other measurements are consistent with a transition of this kind, as are those reported here (figure 4). Then a graded imposition over a 20 in. section of the duct would appear to represent quite closely the way in which lateral straining is initiated. We take

$$\left.\begin{aligned} a(x) &= \frac{a}{2(x_2-x_1)}(x-x_1)^2, \quad \text{for} \quad x_2 > x > x_1, \\ &= ax - \frac{a}{2}(x_1+x_2), \quad \text{for} \quad x > x_2, \end{aligned}\right\} \quad \text{(A 2)}$$

with $x_1 = 10\frac{3}{4}$ in., $x_2 = 30\frac{3}{4}$ in., measured from the cylinder axis, and

$$a = (\log_e 4)/40 = 0{\cdot}0346\,\text{in.}^{-1},$$

the value appropriate to the duct described in § 3 (a), if the boundary-layer compensation is correct.

The results (A 1), (A 2) can be used in conjunction with empirical factors from Townsend's (1956) experiments to construct model scales. In the present experiments distortion began in the range $x/d = 25$ to 160. Then Townsend's data for the range $x/d = 80$ to 160 will provide empirical factors appropriate to this study, since the flow patterns are not strongly dependent on cylinder Reynolds number. For the width of an undistorted wake of self-preserving form we have

$$\frac{b}{d} = \frac{b'}{d^{\frac{1}{2}}}\left(\frac{x-x_0}{d}\right)^{\frac{1}{2}}, \tag{A 3}$$

with $b'/d^{\frac{1}{2}}$ equal to 0·512 for the width at half-peak value of the mean velocity distribution, and 0·732 for the corresponding width of the distribution of the mean square of the longitudinal turbulence component. The virtual origin is given approximately by $x_2 = -20d$. For the peak velocity defect we have in this range

$$\frac{u_m}{U_1} = 1{\cdot}00\left(\frac{x-x_0}{d}\right)^{-\frac{1}{2}}. \tag{A 4}$$

Then the mass and momentum flux defects are characterized in this undistorted self-preserving wake by

$$\frac{b}{d}\frac{u_m}{U_1} = 0{\cdot}512. \tag{A 5}$$

For a wake subjected to the distortion (A 2) and self-preserving throughout, we have from the result (A 1):

$$\left(\frac{b}{b'}\right)^2 = e^{-0{\cdot}00173s^2}\left[34\int_0^{0{\cdot}0294s^2} e^{t^2}\,dt + 14{\cdot}5\right], \tag{A 6}$$

with $s = x - x_1$, for $20 > s > 0$, and

$$\left(\frac{b}{b'}\right)^2 = 20{\cdot}4\,e^{-a(x-x_2)} - 1{\cdot}87e^{-2a(x-x_2)}, \tag{A 7}$$

for $s > 20$, the portion under uniform distortion. Here we have considered the wake behind a $\frac{3}{16}$ in. cylinder in fixing the virtual origin x_0. But note that

$$\left(\frac{b}{b'}\right)^2 \to \frac{e^{-a(x-x')}}{a} \text{ for any } x_0,$$

where x' is a virtual origin depending on the way in which the strain is applied. For the graded distortion (A 2), $x' = \frac{1}{2}(x_1 + x_2)$. For a self-preserving distorted wake we may expect

$$e^{a(s)}\frac{u_m}{U_1}\frac{b}{d} = 0{\cdot}512 \quad \text{with} \quad s = x - x_1, \tag{A 8}$$

corresponding to (A 5). This expression can be used with (A 6), (A 7) to determine the peak velocity defect in the model distorted wakes.

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